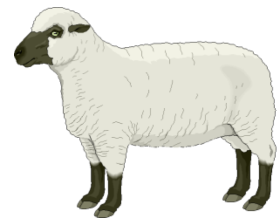
CH 3 – THE RATIONAL NUMBERS



From counting the sheep returning to the cave at day's end, to compressing music and video for transfer across the Internet, mathematics is primarily the study of various sets of numbers.

THE NATURAL NUMBERS

The ***natural numbers*** is the set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. The three dots is a punctuation symbol called an "ellipsis," and means "so on and so on." This set is clearly the fundamental number system in math. All the other numbers systems we shall visit in this course are built upon the natural numbers.

Homework

> If you don't know what $\cup$ and $\subseteq$ mean, see the chapter entitled Sets.

1. a. Is there a smallest element in $\mathbb{N}$?
   b. Is there a largest element in $\mathbb{N}$?

2. How many elements does $\mathbb{N}$ have?

3. Consider the set of *whole numbers*: $\mathbb{W} = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$.
   a. Do $\mathbb{N}$ and $\mathbb{W}$ have the same number of elements?
   b. Which statement is true, $\mathbb{N} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{N}$?

4. T/F:
   a. The sum of two natural numbers is a natural number.
   b. The difference of two natural numbers is a natural number.
   c. The product of two natural numbers is a natural number.
   d. The quotient of two natural numbers is a natural number.

5. Does the equation $7x - 8 = 27$ have a solution in $\mathbb{N}$?

6. Does the equation $3x + 9 = 3$ have a solution in $\mathbb{N}$?

7. Prove each of the following theorems about natural numbers (don't panic — I'll teach you how):

   a. The sum of two even numbers is even.
   b. The product of two even numbers is even.
   c. The sum of two odd numbers is even.
   d. The product of two odd numbers is odd.
   e. The sum of an even and an odd is odd.
   f. The product of an even and an odd is even.
   g. If n is odd, then n^2 is odd.
   h. If n^2 is even, then n is even. This statement follows directly from part g. and one of the concepts from the chapter Logic.

8. a. Find the smallest prime number larger than 50.

   b. The prime factorization of 120 is $2^3 \cdot 3 \cdot 5$. Find the prime factorization of 504.

> "The Natural Numbers were given by God. All the rest were created by man."
>
> –*Kronecker*

❑ *THE INTEGERS*

The ***integers*** (int' e jers) is the set obtained when the natural numbers are combined with zero and the negatives of the natural numbers:

$$\mathbb{Z} = \{1, 2, 3, \ldots\} \cup \{0\} \cup \{\ldots, -3, -2, -1\}$$

In other words,

$$\boxed{\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}}$$

The symbol $\mathbb{Z}$ is used because the German word for number is Zahl.

Recall the homework problem, "Does the equation $3x + 9 = 3$ have a solution in $\mathbb{N}$?" There may be no solution to this equation in $\mathbb{N}$, but there is certainly a solution in the integers $\mathbb{Z}$, namely -2. This is one of the reasons we have the set of integers — we can solve equations that we couldn't otherwise solve.

Homework

9. Write a subset statement involving $\mathbb{N}$ and $\mathbb{Z}$.

10. List the numbers that are in $\mathbb{Z}$ but not in $\mathbb{N}$.

11. T/F:
    a. The sum of two integers is an integer.
    b. The difference of two integers is an integer.
    c. The product of two integers is an integer.
    d. The quotient of two integers is an integer. (You may assume that you're not dividing by zero.)

12. Consider the conjecture:

    If a and b are elements of $\mathbb{Z}$ with $a < b$,
    then there exists a z in $\mathbb{Z}$ such that $a < z < b$.

    First find an example which would make the conjecture true.
    Now give a counterexample to show that the conjecture is false.

13. Does the equation $3x + 9 = 3$ have a solution in $\mathbb{Z}$?

14. Does the equation $2x - 1 = 6$ have a solution in $\mathbb{Z}$?

❑ THE RATIONAL NUMBERS

The equation $x + 2 = 0$ has no solution in $\mathbb{N}$. We remedied that defect by creating $\mathbb{Z}$. But the equation $3x = 4$ has no solution even in $\mathbb{Z}$. We can fix this problem by expanding the number systems even further.

When we write a *ratio* of integers, $\frac{a}{b}$, where $b \neq 0$, we have what's called a ***rational number***. In other words, any number which can be written as a ratio of two integers is called a rational number. Another term for ratio is *quotient*, so we use the letter $\mathbb{Q}$ to represent the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a,b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

> A number is *rational* if it can be written as an integer over an integer, where the bottom, of course, isn't zero. Note that a rational number can be written in an infinite number of ways; for example, the rational number 0.5 can be written as $\frac{1}{2}$, or $\frac{-2}{-4}$, or $\frac{13}{26}$, etc.

EXAMPLES OF RATIONAL NUMBERS:

A. $\frac{7}{8} \in \mathbb{Q}$, since 7 and 8 are integers and $8 \neq 0$.

B. $0 \in \mathbb{Q}$, since 0 can be written $\frac{0}{1}$, among other ways.

C. $-\frac{9}{4} \in \mathbb{Q}$, since $-\frac{9}{4} = \frac{-9}{4}$, which is a ratio of integers.

D. $1.234 \in \mathbb{Q}$, since $1.234 = 1\frac{234}{1000} = \frac{1234}{1000}$, a ratio of integers.

E. $259 \in \mathbb{Q}$ since 259 can be written as the ratio $\frac{259}{1}$.

F. $0.737373... \in \mathbb{Q}$ (recall the three dots mean that the block "73" repeats forever). You will prove that this number is a ratio of integers in the homework.

G. $0.7388888... \in \mathbb{Q}$. The proof is similar to the previous example, but a little trickier.

H. $\sqrt{2}$ is <u>not</u> in $\mathbb{Q}$ (i.e., $\sqrt{2} \notin \mathbb{Q}$). In other words, $\sqrt{2}$ can <u>never</u> be written as the ratio of two integers — it's simply not a fraction consisting of integers. This will be a homework problem in the next chapter which we will prove together in class. The number $\sqrt{2}$ arises as the hypotenuse of a right triangle whose legs are each 1.

I. $\pi \notin \mathbb{Q}$. That is, π cannot be written as a fraction (using integers, of course), but I have no idea how to prove this fact. The number π can be defined as the ratio of the circumference of a circle to its diameter — this ratio is the same regardless of the size of the circle.

Note

The rational numbers are simply the fractions that can be created by using integers (with no zero on the bottom, of course). But every fraction can be converted to a decimal by doing long division. It's also a fact that eventually the decimal <u>must</u> begin repeating in a pattern that will continue forever and ever. For example,

$$\frac{2}{3} = 0.6666\ldots \qquad \text{(the "6" repeats)}$$

$$-\frac{1}{12} = -0.083333\ldots \qquad \text{(the "3" repeats)}$$

$$\frac{2}{7} = 0.285714285714\ldots \qquad \text{(the “285714” repeats)}$$

$$-\frac{9}{2} = -4.50000\ldots \qquad \text{(the “0” repeats)}$$

Therefore, not only can a rational number be defined as a ratio of two integers, but it can also be described as a repeating decimal:

$\mathbb{Q}$ = {all decimals with a *repeating block* of digits}

Final Example

In Example H above we said that $\sqrt{2}$ is not a rational number, and in the Note above we characterized the rational numbers as those numbers whose decimal expansion eventually repeats in a block pattern. Therefore, we can conclude the very important statement that the decimal expansion of $\sqrt{2}$ never begins to repeat in a block pattern that lasts forever. We can also conclude that π is another number that can never be written as a repeating decimal.

> $\frac{43}{99}$ is rational because it can be written as the repeating decimal 0.434343 . . .
>
> But $\sqrt{2}$ and π are <u>not</u> rational because they can <u>never</u> be written as a repeating decimal.

Equations

Recall the homework problem, "Does the equation $2x - 1 = 6$ have a solution in $\mathbb{Z}$?". There may be no solution in $\mathbb{Z}$, but there is a solution in $\mathbb{Q}$; it's $x = \frac{7}{2}$ (or 3.5). This is one of the reasons we have the set of rationals — we can solve equations that we couldn't otherwise solve.

Homework

15. Which of the following numbers are rational?

a. $\frac{5}{19}$ b. $-\frac{12}{5}$ c. 187 d. -14.75 e. 1.222

f. π g. $\sqrt{3}$ h. $\frac{\pi}{\pi}$ i. $\sqrt{100}$ j. $\sqrt{-1}$

k. 2.343434 . . . l. –0.123123123 . . .

m. 2.07932857 . . . [Note: there's no repeating block here]

n. 14.08978865334298

16. What's the importance of the stipulation that $b \neq 0$ in our first definition of $\mathbb{Q}$?

17. Prove that each of the following is rational:

a. $-\frac{2}{3}$ b. -801 c. 0.83 d. 1.125 e. $\frac{7}{16}$

18. Prove that 0.737373. . . $\in \mathbb{Q}$ by showing that this decimal can be written as a fraction (where the top and bottom are integers). [This is a hard problem — it's <u>not</u> cheating in this class to simply check out the full solution given at the end of the chapter.]

19. Prove that 0.5555. . . $\in \mathbb{Q}$ by writing the decimal as a fraction.

20. Explain why $\mathbb{Z} \subseteq \mathbb{Q}$. Is it clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$?

21. a. $\mathbb{N} \cup \mathbb{Q} =$ _____ b. $\mathbb{N} \cap \mathbb{Q} =$ _____

22. a. $\mathbb{Z} \cup \mathbb{Q} =$ _____ b. $\mathbb{Z} \cap \mathbb{Q} =$ _____

23. a. Is $\sqrt{3} \in \mathbb{Q}$? b. Is $\sqrt{49} \in \mathbb{Q}$?

24. Prove that $\frac{3}{7} \in \mathbb{Q}$ by using the decimal definition of a rational number.

25. Explain why the contention that
"π is rational, since it can be written as the ratio of integers, $\frac{22}{7}$"
is pure hokum.

26. T/F: $0.79799799979999\ldots \in \mathbb{Q}$

27. T/F: $0.28288288828888 \in \mathbb{Q}$

28. T/F: The sum of two rational numbers is rational.

29. T/F: The difference of two rational numbers is rational.

30. T/F: The product of two rational numbers is rational.

31. T/F: The quotient of two rational numbers is rational. (You may assume that the divisor is not zero.)

32. Let $q_1, q_2 \in \mathbb{Q}$, where $q_1 < q_2$. Prove that there exists a number $q \in \mathbb{Q}$ such that $q_1 < q < q_2$. Rephrased in English, given any two different rational numbers, show that there is always another rational number somewhere between them. [Note: This property did not hold in $\mathbb{Z}$.]

33. Does the equation $2x - 1 = 6$ have a solution in $\mathbb{Q}$?

34. Does the equation $x^2 - 2 = 0$ have a solution in $\mathbb{Q}$?

35.** Convert $0.7383838\ldots$ to a fraction.

Review Problems

36. Give an element of $\mathbb{Z}$ which is not in $\mathbb{N}$.

37. Prove: If x is even, then x^2 is even.

38. a. The definition of the rationals is $\mathbb{Q} = \{\_\_\_\_ \mid m, n \in \subseteq$, and $\_\_\_\_\_\}$.
    b. Define $\mathbb{Q}$ as a set of decimals.

39. Use the definition (part a. above) to prove that $2.75 \in \mathbb{Q}$.

40. Use the definition (part a. above) to prove that $0.232323\ldots \in \mathbb{Q}$.

41. T/F: For any natural number n, it's always true that $\sqrt{n} \in \mathbb{Q}$.

42. T/F: The difference of two rational numbers is rational.

43. T/F: Let $q_1, q_2 \in \mathbb{Q}$, where $q_1 \neq q_2$. Then there exists a $q \in \mathbb{Q}$ such that q is between q_1 and q_2.

44. Construct an equation which has a solution in $\mathbb{Z}$ but not in $\mathbb{N}$.

45. Construct an equation which has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

46. Give an element of $\mathbb{Q}$ which is not in $\mathbb{Z}$.

47. a. Prove the statement: If x is odd, then x^2 is odd.
    b. State the converse of the statement. Is the converse true?
    c. State the contrapositive of the statement. Is the contrapositive true?

48. a. The definition of the rationals is $\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in$ __, and __ $\neq 0 \}$.

b. What set of numbers consists of all repeating decimals?

49. Use the definition (part a. above) to prove that $0.111 \in \mathbb{Q}$.

50. Use the definition (part a. above) to prove that $9.555\ldots \in \mathbb{Q}$.

51. T/F: There's at least one natural number such that $\sqrt{n} \in \mathbb{Q}$.

52. T/F: The product of two rational numbers is rational.

53. T/F: Let $q_1, q_2 \in \mathbb{Q}$. Then there exists a $q \in \mathbb{Q}$ such that q is between q_1 and q_2.

54. Construct an equation which has a solution in $\mathbb{N}$ but not in $\mathbb{Q}$.

55. Construct an equation which has a solution in $\mathbb{Z}$ but not in $\mathbb{Q}$.

56. True/False:

a. $\pi \in \mathbb{N}$

b. $10^6 \in \mathbb{N}$

c. The product of two natural numbers is a natural number.

d. The difference of two natural numbers is a natural number.

e. The equation $x + 99 = 14$ has a solution in $\mathbb{N}$.

f. $\mathbb{N} \subseteq \mathbb{Z}$

g. The equation $x + 132 = 44$ has a solution in $\mathbb{Z}$.

h. If $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ and $p < q$, then there exists $r \in \mathbb{Z}$ such that $p < r < q$.

i. The number 13/7 is called a rational number.

j. $\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}\}$.

k. $7 \in \mathbb{Q}$.

l. $\sqrt{5} \in \mathbb{Q}$.

m. $-\sqrt{144} \in \mathbb{Q}$

n. $0.29343434... \in \mathbb{Q}$.

o. $2.3874082365139... \in \mathbb{Q}$.

p. If $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ and $p < q$, then there exists $r \in \mathbb{Q}$ such that $p < r < q$.

q. The equation $13n + 17 = -100$ has a solution in $\mathbb{Q}$.

r. The equation $x^2 - 225 = 0$ has its solutions in $\mathbb{Q}$.

s. The equation $y^2 - 125 = 0$ has its solutions in $\mathbb{Q}$.

t. The sum of two odd numbers is odd.

u. $17.09385760225318 \in \mathbb{Q}$.

Solutions

1. a. Yes b. No **2.** ∞ **3.** a. What do you think? b. $\mathbb{N} \subseteq \mathbb{W}$

4. a. T b. F c. T d. F **5.** Yes **6.** No **7.** class exercises

8. a. It's not 51. b. Find primes whose product is 504.

9. $\mathbb{N} \subseteq \mathbb{Z}$ **10.** $\{... -3, -2, -1, 0\}$

11. a. T b. T c. T d. F

12. For the example, take $a = -3$ and $b = 7$. Now choose $0 \in \subseteq$. Note that $-3 < 0 < 7$, as required by the conjecture. As for the counterexample, you're on your own.

13. Yes **14.** No **15.** a, b, c, d, e, h, i, k, l, n

16. Division by zero is undefined.

17. a. $-\frac{2}{3} = \frac{-2}{3}$, a ratio of integers; or, it's equal to –0.6666. . ., a repeating decimal.

b. $-801 = \frac{-801}{1}$, a ratio of integers; or, it's the same as –801.0000. . ., a repeating decimal.

c. $0.83 = \frac{83}{100}$, a ratio of integers; or, it's a repeating decimal 0.83000. . .

d. 1.125 is a repeating decimal (by attaching zeros); or, it can be written as $1\frac{125}{1000} = 1\frac{1}{8} = \frac{9}{8}$, which is a ratio of integers.

e. $\frac{7}{16}$ is a ratio of integers, so it's rational. Also, it can be written as 0.4376, which is a repeating decimal, another indication that it's rational.

18. Let

$$n = 0.737373\ldots$$

Multiply each side of the equation by 100:

$$100n = 73.737373\ldots$$

Subtract the first equation from the second equation:

$$\begin{aligned} 100n &= 73.737373\ldots \\ -\quad n &= 0.737373\ldots \\ \hline 99n &= 73 \\ n &= \frac{73}{99} \end{aligned}$$

Since n stood for the original infinite decimal, and since n has been shown to equal 73/99, it follows that the fraction form of the decimal is $\frac{73}{99}$.

19. Similar to the previous problem, except multiply by 10.

20. Every integer $z \in \mathbb{Z}$ can be written as the fraction $\frac{z}{1} \in \mathbb{Q}$.

21. a. $\mathbb{Q}$ b. $\mathbb{N}$ **22**. a. $\mathbb{Q}$ b. $\mathbb{Z}$ **23**. a. No b. Yes

24. Convert the fraction to a repeating decimal.

25. If π were truly and exactly $\frac{22}{7}$, then π would be rational. The trouble is, π is only approximately equal to that fraction.

26. F **27**. T **28**. T **29**. T **30**. T **31**. T

32. What number lies halfway between the two given numbers?

33. Yes **34**. No

35. $\frac{731}{990}$

36. 0 and -3 are examples of numbers that are in $\mathbb{Z}$ but not in $\mathbb{N}$.

37. If x is even then $x = 2k$, in which case

$x^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even (since it's twice something)
Therefore, x^2 is even.

38. a. $\frac{m}{n}$; $n \neq 0$ b. $\mathbb{Q}$ is the set of all repeating decimals.

39. $2.75 = 2\frac{3}{4} = \frac{11}{4}$, which is the ratio of two integers.

40. Let $n = 0.232323...$ Multiplying both sides by 100 gives
$100n = 23.2323...$ Subtracting the equations gives
$99n = 23$, from whence we deduce that $n = \frac{23}{99}$, which is rational.

41. False; for example, $5 \in \mathbb{N}$, but $\sqrt{5} \notin \mathbb{Q}$.

42. True; Basically, when you subtract two fractions, you get a fraction. Specifically, the difference of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad-bc}{bd}$.

43. True; The midpoint (average) works well, so define $q = \frac{q_1+q_2}{2}$, and we have a rational number which is (halfway) between q_1 and q_2.

44. The solution of the equation $4x + 20 = 16$ is $x = -1$, which is in $\mathbb{Z}$ but not in $\mathbb{N}$.

45. Consider the equation $3x = 7$. Its solution is 7/3, which is rational but not an integer.

46. $\frac{3}{4} \in \mathbb{Q}$, but it's not in $\mathbb{Z}$.

47. a. If x is odd then $x = 2k + 1$, in which case

$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd (because it's 1 more than twice something). Therefore, x^2 is odd.

b. Converse: If x^2 is odd, then x is odd. It's true.

c. Contrapositive: If x^2 is even, then x is even. It's true.

48. a. $\mathbb{Z}$; n b. $\mathbb{Q}$

49. $0.111 = \frac{111}{1000}$, which is the ratio of two integers.

50. Let $n = 9.555...$ Then $10n = 95.555...$ Subtracting gives $9n = 86$, and so $n = \frac{86}{9}$.

51. True; For example if $n = 81$, then $\sqrt{n} = \sqrt{81} = 9 \in \mathbb{Q}$.
Of course, the statement's not true for every natural number.

52. True; The product of the rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ is $\frac{pr}{qs}$, which is rational.

53. False; this is a trick question. The statement does not specify that the two rational numbers must be different from each other. So if we take the two rationals to be the same, say $q_1 = 9$ and $q_2 = 9$, then it is clearly not the case that there's a rational numbers between them. [In fact, there's no number whatsoever that could be between 9 and itself.]

54. This is impossible, since $\mathbb{N} \subseteq \mathbb{Q}$, so how could something be in $\mathbb{N}$ but not in $\mathbb{Q}$?

55. Another impossibility, since $\mathbb{Z} \subseteq \mathbb{Q}$.

56.

a. F	b. T	c. T	d. F	e. F	f. T	g. T	h. F	i. T
j. F	k. T	l. F	m. T	n. T	o. F	p. T	q. T	r. T
s. F	t. F	u. T						

"Education, then, beyond all other devices of human origin, is the great equalizer of the conditions of men – the balance-wheel of the social machinery."

Horace Mann